Choice Probability Generating Functions

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Outline

Motivation

To fully characterize finite-mean RUM

It can be done analogously to how MEV models are introduced in McFadden (1978):

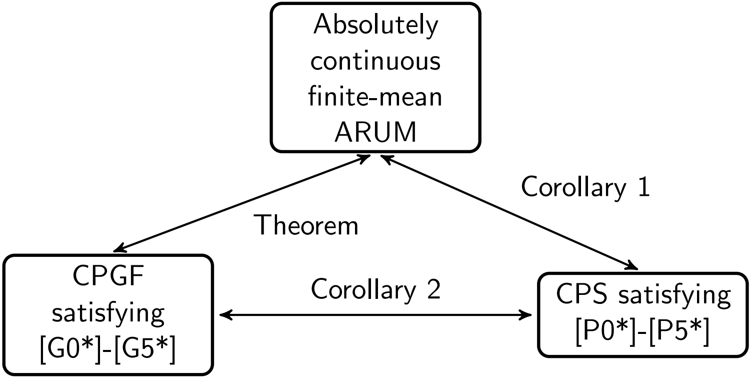
- Define a function G satisfying some conditions
- Then $U = \log G(\mathrm{e}^{V_1}, ..., \mathrm{e}^{V_J}) + \gamma$ satisfies

$$U = E[\max_{j \in \mathcal{C}} U_j]$$

where $\gamma = 0.57721566$ is Euler's constant.

by relaxing some of the conditions of G.

Conclusions



Definitions

Concepts needed:

- Random utility model (RUM)
- Additive random utility model (ARUM)
- Transformed additive random utility model (TARUM)
- Choice probability generating function (CPGF)
- Choice probability system (CPS)

Random utility model

A random utility model (RUM) is a vector $U = (U_1, ..., U_J) \in \mathbb{R}^J$ with CDF $R(u_1, ..., u_J)$. It induces an observable choice probability:

$$P_{\mathcal{C}}(j) = \Pr(U_j > U_k \text{ for } j \neq k \in \mathcal{C}) = \int_{-\infty}^{\infty} \nabla_j R(u, ...u)\, du$$

- Given a continuously differentiable increasing transformation $r : \mathbb{R} \to \mathbb{R}$, the image of a RUM is a RUM with the same associated choice probability.
- This defines a class.
- A representative can be chosen such that $E(U_j)$ is finite $\forall j$

Additive random utility model

Given:

- A finite-mean RUM with CDF $R(u_1, ..., u_J)$ and choice probability $P^0_{\mathcal{C}}(j)$
- $\mathbf{m} = (m_1, ..., m_J) \in \mathbb{R}^J$ a location vector

Then the family,

$$U = (U_1, ..., U_J) = \mathbf{m} + U^0$$

is an additive random utility model (ARUM). Each element is a finite-mean RUM with CDF $R(u_1 - m_1, ..., u_J - m_J)$ and choice probability

$$P_{\mathcal{C}}(j|\mathbf{m}) = \Pr(U_j \geq U_k \text{ for } k \in \mathcal{C}) = \Pr(U^0_j + m_j \geq U^0_k + m_k \text{ for } k \in \mathcal{C})$$

Transformed additive random utility model (TARUM)

Given

- r_j a continuously differentiable increasing transformation with a continuously differentiable inverse,
- $\varepsilon = (\varepsilon_1, ..., \varepsilon_J)$ with an absolutely continuous CDF $F(\varepsilon)$, and
- $\mathbf{m} = (m_1, ..., m_J)$ a location vector.

A finite-mean family of RUM defined implicitly by $\varepsilon_j = r_j(U_j - m_j)$ is a finite-mean transformed additive random utility model (TARUM).

RUM, ARUM, TARUM

Every RUM has a representation that can be embedded in an ARUM with finite mean, or an observationally equivalent TARUM.

Choice probability system (CPS)

A complete choice probability system is a familiy of non-negative functions $P_{\mathcal{C}}(j|\mathbf{m})$ for $j = 1, ..., J$ that sum up to one.

A partial choice probability system is a family of non-negative functions $P_{\mathcal{C}}(j|\mathbf{m})$ defined for $j = 1.,,, k < J$ whose sum does not exceed 1.

Choice probability generating function

A choice probability generating function (CPGF) is a function $G : [0, +\infty]^J \to [0, +\infty]$ with the following properties:

- G1 Weak alterning signs property (ASP)
- G2 Homogeneity
- G3 Boundary
- G4 Integrability

The gradient

$$P_{\mathcal{C}}(j|\mathbf{m}) = \frac{\partial \ln G(\mathrm{e}^{\mathbf{m}})}{\partial m_j}$$

is the choice probability generated by the CPGF G.

Properties [G1]-[G4] are equivalent to another set of properties [G0*]-[G5*]

Theorem

Given

- A finite-mean TARUM $\varepsilon_j = r_j(U_j - m_j)$ with $\varepsilon = (\varepsilon_1, ..., \varepsilon_J)$ with CDF $F(\varepsilon)$
- An observationally equivalent ARUM $U_j = m_j + \zeta_j$ with $\zeta = (\zeta_1, ..., \zeta_J)$ with CDF $R(\zeta) \equiv F(r(U - \mathbf{m}))$

Then, an associated CPGF exists and is defined by:

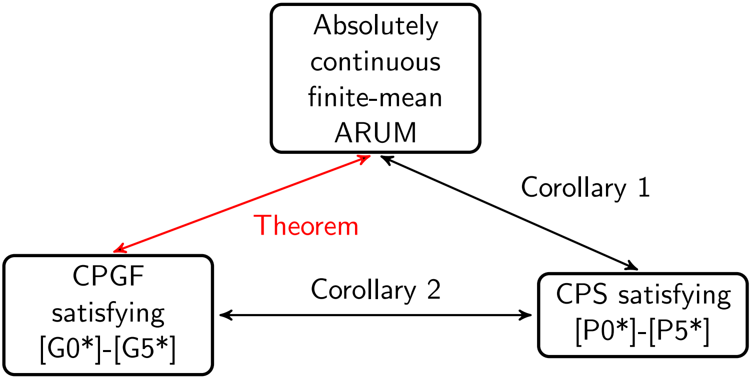
$$\ln G(\mathrm{e}^{\mathbf{m}}) \equiv E[\max_{j \in \mathcal{C}} U_j] \equiv \int_0^{+\infty} [1 - R(u - \mathbf{m})]\, du - \int_{-\infty}^{0} R(u - \mathbf{m})\, du$$

The choice probability implied by the TARUM satisfies:

$$P_{\mathcal{C}}(j|\mathbf{m}) = \Pr(U_j > U_k \text{ for } k \neq j) = \partial \ln G(\mathrm{e}^{\mathbf{m}}) / \partial m_j$$

The reciprocal is also true.

Theorem


Absolutely continuous finite-mean ARUM
Theorem
Corollary 1
CPGF satisfying [G0*]-[G5*]
Corollary 2
CPS satisfying [P0*]-[P5*]

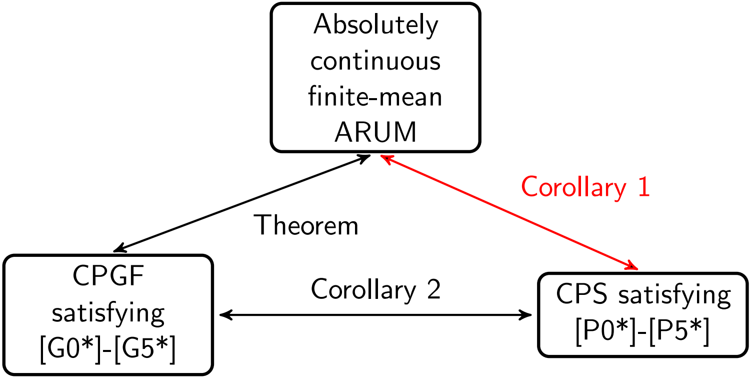
Corollary 1

A complete or partial CPS is ARUM-consistent if and only if it satisfies:

- P1 Alterning signs property (ASP)
- P2 Homogeneity
- P3 Boundary
- P4 Integrability

These can be expressed as another set of properties [P0*]-[P5*]

Corollary 1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Corollary 2

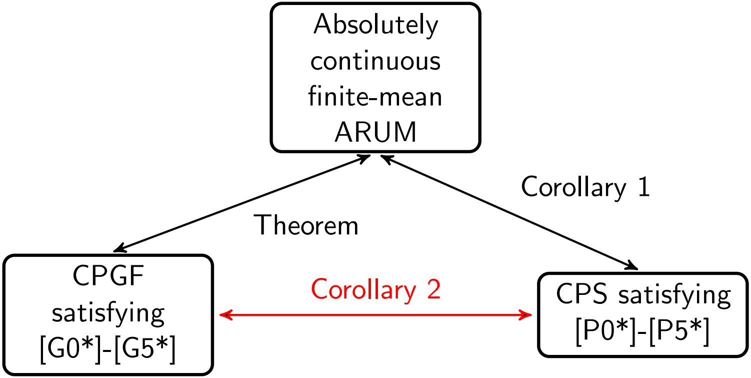
If $P_{\mathcal{C}}(1|\mathbf{m})$ is the first probability in a CPS satisfying [P0*]-[P5*] then

$$h(\mathbf{m}) = m_1 + \int_{m_1}^{+\infty} [1 - P(1|v_1, \mathbf{m}_{-1})]$$

is a CPGF satisfying [G0*]-[G5*] and whose gradient gives the partial CPS.

Notation: Given $\mathbf{m} = (m_1, ..., m_J)$, $\mathbf{m}_{-j} = (m_1, ..., m_{j-1}, m_{j+1}, ..., m_J)$ denotes a vector excluding component j.

Corollary 2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Choice probability from a single alternative is enough

A choice probability $P_{\mathcal{C}}(1|\mathbf{m})$ for a single alternative that satisfies [P0*]-[P5*] (for consistency with an ARUM) can be used to determine the CPGF from which the choice probabilities for the rest of the alternatives can be determined.

Simple example: logit model

Notation:

$$h(\mathbf{m}) = \ln G(\mathrm{e}^{\mathbf{m}})$$

Corollary 2:

$$h(\mathbf{m}) = m_1 + \int_{m_1}^{+\infty} [1 - P(1|v_1, \mathbf{m}_{-1})]$$

Multinomial logit:

$$P_{\mathcal{C}}(1|\mathbf{m}) = \frac{\mathrm{e}^{m_1}}{\sum_{j=1}^{J} \mathrm{e}^{m_j}}$$

Simple example: logit model

From one probability we can recover $h(\mathbf{m})$:

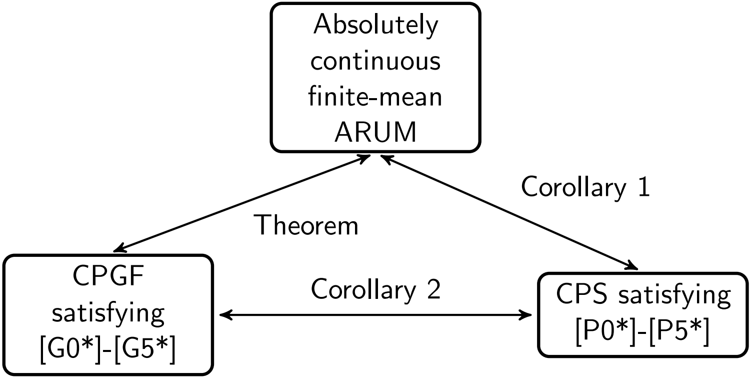
$$\begin{aligned}
h(\mathbf{m}) &= m_1 + \int_{\mathbf{m}_1}^{+\infty} 1 - P(1|v_1, m_{-1})\, dv_1 \\
&= m_1 + \int_{m_1}^{+\infty} 1 - \frac{\mathrm{e}^{v_1}}{\mathrm{e}^{v_1} + \sum_{j=2}^{J} \mathrm{e}^{m_j}}\, dv_1 \\
&= m_1 + \left[v_1 - \ln\left(\mathrm{e}^{v_1} + \sum_{j=2}^{J} \mathrm{e}^{m_j} \right) \right]_{v_1=m_1}^{v_1=+\infty} \\
&= m_1 - m_1 + \ln\left(\mathrm{e}^{m_1} + \sum_{j=2}^{J} \mathrm{e}^{m_j} \right) \\
&= \ln\left(\sum_{j=1}^{J} \mathrm{e}^{m_j} \right)
\end{aligned}$$

Simple example: logit model

And from $h(\mathbf{m})$ the rest of probabilities:

$$P_{\mathcal{C}}(i|\mathbf{m}) = \frac{\partial}{\partial m_i} h(\mathbf{m}) = \frac{\partial}{\partial m_i} \ln \left(\sum_{j=1}^{J} \mathrm{e}^{m_j} \right) = \frac{\mathrm{e}^{m_i}}{\sum_{j=1}^{J} \mathrm{e}^{m_j}}$$

Conclusions



Future work

To give an application to this theoretical work. But... how?

- To use the CPGF approach to propose new models
- To take advantage of the fact that full information on choice probabilities is contained in only one of them:
  1. Specify the choice probability for one alternative
  2. Computation of the CPGF and its gradient.
  3. This CPGF could then be used to calculate willingness to pay (WTP) for a policy change

Thanks for your attention!

Questions?

APPENDIX

Properties [G1]-[G4]

[G1] Weak ASP

In $G(y)$ satisfies the ASP, so that for any permutation σ of $(1, ..., J)$ and $k = 1, ..., J$, its mixed partial derivatives are independent of the order of differentiation and

$$\chi_{\sigma:k}(\ln y) \equiv (-1)^k \nabla_{\sigma:k} \ln G(y) \geq 0$$

[G2] Homogeneity

For each $\lambda > 0$ and $y \in [0, +\infty]^J$, $G(\lambda y) = \lambda G(y)$

Properties [G1]-[G4]

[G3] Boundary
G(0)=0, and for $j = 1, ..., J$, if $1_{(j)}$ denotes a unit vector with the jth component equal to one, then $\lim_{\lambda\to\infty} G(\lambda 1_{(j)}) \to \infty$

[G4] Integrability
Let $\mathbf{m} = (m_1, ..., m_J)$, $\mathrm{e}^{\mathbf{m}} = (\mathrm{e}^{m_1}, ..., \mathrm{e}^{m_J})$, $L = \{\mathbf{m} \in \mathbb{R}^J | \sum_{j=1}^J m_j = 0\}$.
Then the following holds:

$$\int_L \chi_{1,...,J}(\mathbf{m})\, d\mathbf{m} = J^{-1}$$

$$\int_L |m_j| \chi_{1,...,J}(\mathbf{m})\, d\mathbf{m} < +\infty$$

$$\max_{k\neq j} m_k - m_j \to \infty \implies \chi_j(\mathbf{m}) \to 0$$

Properties [G0*]-[G5*]

Let $h(\mathbf{m}) = \ln G(\mathrm{e}^{\mathbf{m}})$, then [G1]-[G4] are equivalent to [G0*]-[G5*], where:

[G0*] $h(\mathbf{m})$ is a convex function whose mixed partial derivatives exist

[G1*] $(-1)^{k-1} h_{j_1,\ldots,j_k}(\mathbf{m}) \geq 0$ for any $k = 1, \ldots, J$ and distinct indices $j_1, \ldots, j_k$

[G2*] $h(\mathbf{m} - \gamma) = h(\mathbf{m}) - \gamma$ for any scalar γ

Properties [G0*]-[G5*]

[G3*] $\lim_{m_j \to -\infty} \frac{h(\mathbf{m})}{m_j} = 0$,
$\lim_{m_j \to -\infty} m_j h_j(\mathbf{m}) = 0$,
$\lim_{m_j \to +\infty} \frac{h(\mathbf{m})}{m_j} = 1$,
$\lim_{m_j \to +\infty} m_j [1 - h_j(\mathbf{m})] = 0$,
$\int_{v=-\infty}^{0} h_j(m_j, \mathbf{m_{-j}})\, dm_j < +\infty$, and
$\int_{m_j=0}^{+\infty} [1 - h_j(m_j, \mathbf{m}_{-j})]\, dm_j < +\infty$.

Properties [G0*]-[G5*]

[G4*] $(-1)^{J-1} \int_{\mathbf{m_{-j}}=-\infty}^{+\infty} \frac{\partial^J h(\mathbf{m})}{\partial m_1 \ldots \partial m_J} \, d\mathbf{m_{-j}} = 1$ for each m_j

[G5*]
$- \int_{m_k=-\infty}^{+\infty} |m_k| h_{jk}(-\infty, \ldots, -\infty, m_j, -\infty, \ldots, -\infty, m_k, -\infty, \ldots, -\infty) \, dm_k < +\infty$ for each m_j and $k \neq j$

Properties [P0*]-[P5*]

[P0*]

- Mixed partial derivatives of $P_{\mathcal{C}}(j|\mathbf{m})$ with respect to $\mathbf{m}$ exist
- $P_{\mathcal{C}}(j|\mathbf{m})$ is monotone, non-decreasing in m_j and monotone non-increasing in $\mathbf{m}_{-j}$
- $\partial P_{\mathcal{C}}(j|\mathbf{m})/\partial m_i = \partial P_{\mathcal{C}}(i|\mathbf{m})/\partial m_j$ for $1 \leq i\,,\, j \leq k$

[P1*] $(-1)^n \partial^n P_{\mathcal{C}}(j_0|\mathbf{m})/\partial m_{j_1} ... \partial m_{j_n} \geq 0$ for any n and distinct indices $j_0, ..., j_k$, with $j_0 \leq k$

[P2*] $P_{\mathcal{C}}(j|\mathbf{m} - \gamma) \equiv P_{\mathcal{C}}(j|\mathbf{m})$ for any scalar γ

Properties [P0*]-[P5*]

[P3*] $\lim_{m_j \to -\infty} m_j P_{\mathcal{C}}(j|\mathbf{m}) = 0$,
$\lim_{m_j \to +\infty} m_j [1 - P_{\mathcal{C}}(j|\mathbf{m})] = 0$,
$\int_{-\infty}^{0} P_{\mathcal{C}}(j|m_j, \mathbf{m}_{-j})\, dm_j < +\infty$, and
$\int_{0}^{+\infty} [1 - P_{\mathcal{C}}(j|m_j, \mathbf{m}_{-j})]\, dm_j < +\infty$

[P4*] $(-1)^{J-1} \int_{\mathbf{m_{-j}}=-\infty}^{+\infty} \frac{\partial^{J-1} P_{\mathcal{C}}(j|\mathbf{m})}{\partial m_1 \ldots \partial m_{j-1} \partial m_{j+1} \ldots \partial m_J}\, d\mathbf{m_{-j}} = 1$ for each m_j

[P5*] $(-1)^{J-1} \int_{m_i=-\infty}^{+\infty} |m_i| \frac{\partial P_{\mathcal{C}}(j|\mathbf{m})}{\partial m_k}\, dm_i < +\infty$ for each m_j and $k \neq j$

Definition: ARUM-consistent

A complete or partial CPS is ARUM-consistent if there exists an ARUM with CDF F such that for $1 \leq j \leq k \leq J$,

$$\begin{aligned} P_{\mathcal{C}}(j|\mathbf{m}) &= \text{Prob}(\mathbf{U}|U_j + m_j \geq U_k + m_k \text{ for } k \neq j) \\ &\equiv \int_{-\infty}^{+\infty} F_j(v - m_1, \ldots, v - m_j)\, dv \end{aligned}$$